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Normalized Eigenvectors for Nonlinear Abstract and Elliptic Operators*

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We consider the eigenvalue problem

$$Au - C(\lambda, u) = 0$$

for nonlinear operators A, C in a Banach space. The operator A is of type $(S)_+$, or m-accretive, or maximal monotone, while the operator C is compact, or just continuous and bounded, whenever the resolvents of A are compact. The eigenvalues are not necessarily of multiplicative nature (as in the problem $Au - \lambda Cu = 0$). Methods are introduced with applications to nonlinear eigenvalue problems involving the existence of eigenvectors with new normalized conditions. Most of the results are new even in the special case of $Au - \lambda Cu = 0$.

Key Words: maximal monotone operator; m-accretive operator; existence of eigenvalues; normalized eigenvectors; elliptic problems.

1. INTRODUCTION AND PRELIMINARIES

In what follows, X, Y are real Banach spaces. We denote their norms by $\|\cdot\|$. We denote by J the normalized duality mapping of X. The symbols $\langle x^*, x\rangle$ and $\langle x, x^*\rangle$ denote the value of the functional $x^* \in X^*$ at $x \in X$. We denote by $D(T)$, $R(T)$, and $G(T)$ the effective domain, the range, and the graph of a mapping $T: X \to 2^Y$, respectively. We have $D(T) = \{x \in X : Tx \neq \emptyset\}$, $R(T) = \bigcup_{x \in D(T)} Tx$ and $G(T) = \{(x, y) : x \in D(T),\ y \in Tx\}$.

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In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence. The symbol $\mathscr{R}$ ($\mathscr{R}_+$) stands for the set $(-\infty, \infty)$ ($(0, \infty)$) and the symbols ∂D, $\operatorname{int} D$, and $\bar{D}$ denote the strong boundary, interior, and closure of the set D, respectively. An operator $T: X \supset D(T) \to Y$ is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is called "demicontinuous" if it is strong-weak continuous on $D(T)$. It is "compact" if it is continuous and maps bounded sets onto relatively compact sets. It is "α-homogeneous" if $x \in D(T)$ implies $tx \in D(T)$ and $T(tx) = t^\alpha Tx$, $x \in D(T)$, $t > 0$, where α is a fixed positive constant.

An operator $T: X \supset D(T) \to 2^{X^*}$ is "monotone" if

$$\langle u^* - v^*, x - y \rangle \geqslant 0 \text{ for every } x, y \in D(T) \text{ and every } u^* \in Tx,\ v^* \in Ty. \quad (*)$$

A monotone operator T is "strongly monotone" if 0 in the right-hand side of $(*)$ is replaced by $\alpha \|x - y\|^2$, where $\alpha > 0$ is a fixed number. A monotone operator T is called "maximal monotone" if its graph $G(T)$ is a maximal subset of $X \times X^*$. A monotone operator T is maximal monotone if and only if $R(T + \lambda J) = X^*$ for every $\lambda > 0$. If X and X^* are locally uniformly convex and X is reflexive, then $J: X \to X^*$ is a bounded, bicontinuous, and surjective mapping which satisfies condition $(\mathrm{S})_+$ (see Definition 1 below).

An operator $T: X \supset D(T) \to 2^X$ is "accretive" if for every $x, y \in D(T)$ and every $u \in Tx$, $v \in Ty$ there exists $j \in J(x - y)$ such that

$$\langle Tx - Ty, j \rangle \geqslant 0.$$

An accretive operator is "m-accretive" if $R(T + \lambda I) = X$ for all $\lambda > 0$. An accretive operator is "strongly accretive" if zero in the above inequality is replaced by $\alpha \|x - y\|^2$, where $\alpha > 0$ is independent of $x, y \in D(T)$.

In this paper we study eigenvalue problems for abstract operators and give several applications to nonlinear elliptic equations.

Eigenvalue problems of the type

$$Au - \lambda Cu = 0 \tag{1}$$

have been studied, for example, by Krasnosel'skij [5] (where A is the identity operator), Skrypnik [6] and [7] (where A is of type $(\mathrm{S})_+$), Fitzpatrick and Petryshyn [2] (for A-proper operators), Zvyagin [9] (for Fredholm mappings), Guan and Kartsatos [3], and Kartsatos [4] (for m-accretive and maximal monotone operators).

These papers were based on degree theories for various classes of operators, namely, the Leray–Schauder degree theory for compact displacements of the identity, Skrypnik's degree theory for operators of type $(\mathrm{S})_+$, or the Browder–Petryshyn degree theory for A-proper mappings.

Applications of various results in the papers [2–4, 6, 7, 9] involve the existence of eigenvectors belonging to the closure of a bounded open set (without normalized conditions), or the boundary of a bounded open set in the energy space with normalized conditions of special forms.

In order to clarify the situation, we consider, for example, the eigenvalue problem

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j\left(x, \frac{\partial u}{\partial x}\right) &= \lambda C(x, u), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{2}$$

where Ω is an open bounded subset of $\mathscr{R}^n$ with sufficiently smooth boundary. By imposing known conditions on the coefficients, it is possible to consider this problem in the energy space $W_0^{1,m}(\Omega)$, for some $m > 1$. Then, under some further conditions, it is possible to establish the existence of eigenvectors satisfying a normalized condition of the form

$$F(u) \equiv \int_\Omega f\left(x, u, \frac{\partial u}{\partial x}\right) dx = 1,$$

where the functional F is such that

$$F(u) \to \infty, \qquad \text{for} \quad \|u\|_{W_0^{1,m}(\Omega)} \to \infty.$$

This problem is analogous to the problem of finding eigenvectors of the differential problem (2) belonging to the boundary of an open and bounded set D in $W_0^{1,m}(\Omega)$. However, in principle, it is quite a different problem to find eigenvectors of the differential problem (2) satisfying the normalized condition

$$G(u) \equiv \int_\Omega g(x, u)\, dx = 1,$$

or lying on the boundary of a set which is actually unbounded in the energy space. The solvability of such a problems is possible by reducing the differential problems to operator equations with densely defined (and usually unbounded) operators.

In this paper we develop methods for studying the eigenvalue problem

$$Au - C(\lambda, u) = 0, \tag{3}$$

which are also new for the problem (1). In the case of densely defined operators A, we consider always unbounded operators. We also introduce new types of conditions which are tied to the unboundedness of the operator A, or the behavior of the operator C as $\lambda \to \infty$. Condition (A_∞) on the operator A (Definition 3, Section 3) and condition (i) on the operator C in Lemma 1 are examples of such considerations.

The paper is organized as follows. In Section 2 we give a special extension lemma for the operator $C(\lambda, u)$ which provides a basis for the study of the problem (3). In Section 3 we give results on eigenvalue problems for the operator equation (3) in the following cases:

— the operator A is of type $(\mathrm{S})_+$ and C is compact (Theorem 1);

— the operator A is maximal monotone and C is compact (Theorem 2);

— the operator A is m-accretive and C is compact (Theorem 3);

— the operator A is maximal monotone with compact resolvents and C is continuous and bounded (Theorem 4);

— the operator A is m-accretive with compact resolvents and C is continuous and bounded (Theorem 5).

In Section 4 we study three types of problems for nonlinear elliptic equations:

1. nonlinear equations in nondivergence form;

2. higher order equations in divergence form with energy space $W_0^{m,2}(\Omega)$ and normalized condition $\|u\|_{W^{m-1,2}(\Omega)} = 1$;

3. second order divergence form equations (2) with energy space $W_0^{1,m}(\Omega)$ and normalized conditions of the type $\|u\|_{L^2(\Omega)} = 1$.

Finally, in Section 5 we give a result where the perturbation C is defined only on the domain of the operator A.

We hope that the methods of this paper will provide new directions in the study of the solvability of problems involving nonlinear elliptic operators under degenerate conditions. By our approach, these problems can be reduced to problems involving operator equations with densely defined unbounded operators.

2. EXTENSION LEMMAS

In what follows, X_1, X_2 are real infinite dimensional Banach spaces with norms $\|\cdot\|_1$, $\|\cdot\|_2$, respectively. The symbol $\mathscr{D}$ stands for a bounded open set in X_1, or any other Banach space under consideration.

Lemma 1. *Assume that* $C\colon \overline{\mathscr{R}_+} \times \overline{\mathscr{D}} \to X_2$ *is continuous and satisfies the following conditions*:

(i) *there exists a positive number* $\mathcal{N}$ *such that the closure of the set*

$$E=\left\{\frac{C(\lambda, u)}{\|C(\lambda, u)\|_2}: u \in \bar{\mathscr{D}}, \lambda \in \overline{\mathscr{R}_+}, \|C(\lambda, u)\|_2 \geqslant \mathcal{N}\right\} \tag{4}$$

is not equal to $S_1=\{v \in X_2 : \|v\|_2=1\}$;

(ii) *the following property holds*:

$$\lim_{\lambda \to \infty} m_\lambda = +\infty, \qquad \textit{where} \quad m_\lambda = \inf\{\|C(\lambda, u)\|_2 : u \in \partial\mathscr{D}\}. \tag{5}$$

Then there exists a continuous operator $\tilde{C}(\lambda, u)$: $\overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to X_2$ *which has the properties*

$$\tilde{C}(\lambda, u) = C(\lambda, u), \quad (\lambda, u) \in \overline{\mathscr{R}_+} \times \partial\mathscr{D}; \tag{6}$$

$$\lim_{\lambda \to \infty} \mu_\lambda = +\infty, \qquad \textit{where} \quad \mu_\lambda = \inf\{\|\tilde{C}(\lambda, u)\|_2 : u \in \bar{\mathscr{D}}\}. \tag{7}$$

Proof. Let ϕ: $\overline{\mathscr{R}_+} \to \overline{\mathscr{R}_+}$ be a continuous function which equals 1 on $[1, \infty)$, 0 on $[0, 1/2]$ and is such that $\phi(s) \in (0, 1)$ for $s \in (1/2, 1)$. Let $K=\bar{E}$, where E is the set defined by (4). By assumption (i), there exists $\eta \in X_2$ such that $\|\eta\|_2 = 1$, $\eta \notin K$ and

$$\delta = \inf\{\|\tau\eta - z\|_2 : z \in K, 0 \leqslant \tau \leqslant 2\} > 0. \tag{8}$$

Define the operator $\tilde{C}$: $\overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to X_2$ by

$$\tilde{C}(\lambda, u) = \psi(\lambda, u)\, C(\lambda, u) - m_\lambda[1 - \psi(\lambda, u)]\, \eta, \tag{9}$$

where m_λ is defined by (5) and

$$\psi(\lambda, u) = \phi((1+m_\lambda)^{-1}(1 + \|C(\lambda, u)\|_2)).$$

Evidently, the operator $\tilde{C}$, defined by (9), is continuous. If $u_0 \in \partial\mathscr{D}$, then from (5) we have $\|C(\lambda, u_0)\|_2 \geqslant m_\lambda$, for every $\lambda \in \overline{\mathscr{R}_+}$. By the definition of the functions $\phi(s)$, $\psi(\lambda, u)$, we obtain $\psi(\lambda, u_0) = 1$ and, consequently, the validity of the equality (6).

We now prove property (7) for the operator $\tilde{C}$. To this end, we assume that λ satisfies

$$m_\lambda \geqslant 2(\mathcal{N} + 1). \tag{10}$$

Let u be an arbitrary point in $\bar{\mathscr{D}}$. We shall consider three possibilities,

$$\|C(\lambda, u)\|_2 \geqslant m_\lambda, \tag{11}$$

$$\|C(\lambda, u)\|_2 \leqslant \frac{m_\lambda - 1}{2}, \tag{12}$$

$$\frac{m_\lambda - 1}{2} < \|C(\lambda, u)\|_2 < m_\lambda, \tag{13}$$

for a fixed number λ satisfying the inequality (10).

If (11) holds, then $\psi(\lambda, u) = 1$ and from (9) we have

$$\|\tilde{C}(\lambda, u)\|_2 = \|C(\lambda, u)\|_2 \geqslant m_\lambda. \tag{14}$$

If (12) holds, then $\psi(\lambda, u) = 0$ and from (9) we have

$$\|\tilde{C}(\lambda, u)\|_2 = m_\lambda \|\eta\|_2 = m_\lambda. \tag{15}$$

In the case of (13) we consider two possibilities:

$$m_\lambda[1 - \psi(\lambda, u)] \leqslant 2 \|C(\lambda, u)\|_2 \cdot \psi(\lambda, u), \tag{16}$$

$$m_\lambda[1 - \psi(\lambda, u)] > 2 \|C(\lambda, u)\|_2 \cdot \psi(\lambda, u). \tag{17}$$

For λ, u satisfying the estimate (16), we have

$$\psi(\lambda, u) \geqslant [m_\lambda + 2 \|C(\lambda, u)\|_2]^{-1} \cdot m_\lambda.$$

From this inequality and (8) we derive

$$\begin{aligned} \|\tilde{C}(\lambda, u)\|_2 &= \psi(\lambda, u) \|C(\lambda, u)\|_2 \cdot \left\| \frac{C(\lambda, u)}{\|C(\lambda, u)\|_2} - \frac{m_\lambda[1 - \psi(\lambda, u)]\, \eta}{\psi(\lambda, u) \|C(\lambda, u)\|_2} \right\|_2 \\ &\geqslant \delta \frac{\|C(\lambda, u)\|_2\, m_\lambda}{m_\lambda + 2 \|C(\lambda, u)\|_2} \\ &\geqslant \frac{m_\lambda - 1}{6}\, \delta, \end{aligned} \tag{18}$$

for every $(\lambda, u) \in \overline{\mathscr{R}_+} \times \bar{\mathscr{D}}$ satisfying the inequalities (13) and (16).

If the inequality (17) is satisfied, then we have

$$\psi(\lambda, u) < [m_\lambda + 2 \|C(\lambda, u)\|_2]^{-1} \cdot m_\lambda.$$

From this estimate we obtain the inequality

$$\begin{aligned}\|\tilde{C}(\lambda, u)\|_2 &= m_\lambda[1-\psi(\lambda, u)] \left\| \eta - \frac{\psi(\lambda, u)\ \|C(\lambda, u)\|_2}{m_\lambda[1-\psi(\lambda, u)]} \cdot \frac{C(\lambda, u)}{\|C(\lambda, u)\|_2} \right\|_2 \\ &\geqslant \frac{1}{2} m_\lambda[1-\psi(\lambda, u)] \\ &\geqslant \frac{\|C(\lambda, u)\|_2 \cdot m_\lambda}{m_\lambda + 2\ \|C(\lambda, u)\|_2} \\ &\geqslant \frac{m_\lambda - 1}{6},\end{aligned}$$

for $(\lambda, u) \in \overline{\mathscr{R}_+} \times \bar{\mathscr{D}}$ satisfying the conditions (13) and (17).

From this and the inequalities (14), (15) and (18) follows the estimate

$$\|\tilde{C}(\lambda, u)\|_2 \geqslant \tfrac{1}{6}(m_\lambda - 1)\,\delta,$$

for arbitrary $(\lambda, u) \in \overline{\mathscr{R}_+} \times \bar{\mathscr{D}}$. Using the last estimate and (5), (8), we obtain that the operator $\tilde{C}$ satisfies the inequality (7), which completes the proof of the lemma. ∎

Remark 1. From the definition of the operator $\tilde{C}$ it follows immediately that if C is a compact operator, then so is $\tilde{C}$.

Remark 2. If the operator C satisfies the condition $C(0, u) \equiv 0$, $u \in \partial\mathscr{D}$, then the same is true for the operator $\tilde{C}$. This follows from (6).

Remark 3. If the operator C in Lemma 1 is such that the closure of the set E, defined by (4), is compact, then there exists a number $\tilde{\Lambda} > 0$ such that the closure of the set

$$\tilde{E} = \left\{ \frac{\tilde{C}(\lambda, u)}{\|\tilde{C}(\lambda, u)\|_2} : \lambda \geqslant \tilde{\Lambda},\ u \in \bar{\mathscr{D}} \right\} \tag{19}$$

is also compact. This assertion follows from the construction of the operator $\tilde{C}$ with

$$\tilde{\Lambda} = \sup\{\lambda \in \overline{\mathscr{R}_+} : m_\lambda \leqslant 2\mathscr{N} + 1\}.$$

LEMMA 2. *Assume that the operator C satisfies all the conditions of Lemma* 1. *Assume, further, that the weak closure of the set E, defined by* (4), *does not contain zero. Then there exists an operator $\tilde{C}$ which satisfies all the conditions of Lemma* 1 *and is such that the weak closure of the set $\tilde{E}$ in* (19) *does not contain zero.*

Proof. Let $\bar{E}^w$ be the weak closure of the set E and define the set

$$E' = \left\{ \frac{v}{\|v\|_2} : v \in \bar{E}^w \right\}.$$

From our assumptions it follows that the set E' is closed and such that $E' \neq S_1$. Thus, we can choose $\eta' \in S_1$ such that $\eta' \notin E'$ and

$$\delta' = \inf\{ \|\tau\eta' - z\|_2 : \tau \in \overline{\mathscr{R}_+}, z \in E' \} > 0.$$

We now construct the operator $\tilde{C}$ by means of the formula

$$\tilde{C}(\lambda, u) = \psi(\lambda, u)\, C(\lambda, u) - m_\lambda[1 - \psi(\lambda, u)]\, \eta'$$

with the same function $\psi(\lambda, u)$ as in (9). It is evident that this operator $\tilde{C}$ satisfies all the conditions of Lemma 1. We only need to check that the weak closure of the set $\tilde{E}$ does not contain zero.

Let $(\lambda_n, u_n) \in [\tilde{\Lambda}, \infty) \times \bar{\mathscr{D}}$ be such that

$$\frac{\tilde{C}(\lambda_n, u_n)}{\|\tilde{C}(\lambda_n, u_n)\|_2} \rightharpoonup \xi, \tag{20}$$

for some $\xi \in X_2$. If λ_n, u_n satisfy one of the inequalities (11), (12), then, clearly, $\xi \neq 0$. Let now λ_n, u_n be such that

$$\tfrac{1}{2}(m_{\lambda_n} - 1) \leqslant \|C(\lambda_n, u_n)\|_2 \leqslant m_{\lambda_n}.$$

Then, as in the proof of Lemma 1, we have the estimate

$$\|\tilde{C}(\lambda_n, u_n)\| \geqslant \tfrac{1}{6}(m_{\lambda_n} - 1)\, \delta'.$$

Consequently, passing to subsequences if necessary, we may assume the existence of the limits

$$\frac{\psi(\lambda_n, u_n)\, \|C(\lambda_n, u_n)\|_2}{\|\tilde{C}(\lambda_n, u_n)\|_2} \to \theta_1, \qquad \frac{m_{\lambda_n}[1 - \psi(\lambda_n, u_n)]}{\|\tilde{C}(\lambda_n, u_n)\|_2} \to \theta_2,$$

$$\|C(\lambda_n, u_n)\|_2^{-1}\, C(\lambda_n, u_n) \rightharpoonup \zeta, \tag{21}$$

where the nonnegative numbers θ_1, θ_2 satisfy $\theta_1 + \theta_2 > 0$ and $\zeta \in X_2$ with $\zeta \neq 0$. Using the definition of the operator $\tilde{C}$ and the limits in (20) and (21), we obtain the equality

$$\xi = \theta_1 \zeta - \theta_2 \eta'.$$

Taking into consideration that $\zeta \neq 0$, $\theta_1 + \theta_2 > 0$ and the fact that $\delta' > 0$, we obtain $\xi \neq 0$, which finishes the proof. ∎

3. EXISTENCE OF EIGENVECTORS FOR ABSTRACT OPERATORS

This section is devoted to the existence of eigenvalues for nonlinear operators. We start with principal operators satisfying condition $(S)_+$. We adopt the notation for the degree function $Deg(\cdot, \cdot, \cdot)$ for such mappings from [7, 8].

DEFINITION 1. The operator $A: D(A) \subset X \to X^*$ satisfies condition $(S)_+$ if for every sequence $\{u_n\} \subset D(A)$ such that $u_n \rightharpoonup u$ and

$$\limsup_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leqslant 0$$

we have $u_n \to u_0$.

THEOREM 1. *Let X be reflexive and let $\mathscr{D}$ be a bounded open set in X. Assume that $A: \overline{\mathscr{D}} \to X^*$ is a bounded demicontinuous operator which satisfies condition $(S)_+$ and is such that $Au \neq 0$, $u \in \partial\mathscr{D}$, and $Deg(A, \overline{\mathscr{D}}, 0) \neq 0$. Let $C: \overline{\mathscr{R}_+} \times \overline{\mathscr{D}} \to X^*$ be a compact operator which satisfies conditions* (i), (ii) *of Lemma* 1 *(with $X_2 = X^*$) and is such that $C(0, u) \equiv 0$, for $u \in \partial\mathscr{D}$. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that*

$$Au_0 - C(\lambda_0, u_0) = 0. \tag{22}$$

Proof. At first we show the existence of $\tilde{\lambda} \in \mathscr{R}_+$ such that

$$Deg(A - \tilde{C}(\lambda, \cdot), \overline{\mathscr{D}}, 0) = 0, \qquad \text{for} \quad \lambda \geqslant \tilde{\lambda}, \tag{23}$$

where $\tilde{C}: \overline{\mathscr{R}_+} \times \overline{\mathscr{D}} \to X^*$ is the operator constructed by Lemma 1. The above degree is well defined because, by Remark 1, we have that the operator $A - \tilde{C}(\lambda, \cdot): \overline{\mathscr{D}} \to X^*$ satisfies condition $(S)_+$ for every $\lambda \in \overline{\mathscr{R}_+}$. If for a given positive number λ and every $u \in \partial\mathscr{D}$ we have $Au - \tilde{C}(\lambda, u) \neq 0$, then the degree of the mapping $A - \tilde{C}(\lambda, \cdot): \overline{\mathscr{D}} \to X^*$ is well defined.

Assume that (23) is false. Then we have two possibilities:

(α) there exist sequences $\{\lambda_n'\}$, $\{u_n'\}$ such that $\lambda_n' \to \infty$, $\{u_n'\} \subset \partial\mathscr{D}$ and

$$Au_n' - \tilde{C}(\lambda_n', u_n') = 0;$$

(β) there exists a sequence λ_n'' such that $\lambda_n'' \to \infty$, $Au - \tilde{C}(\lambda_n'', u) \neq 0$, $u \in \partial\mathscr{D}$, and $Deg(A - \tilde{C}(\lambda_n'', \cdot), \overline{\mathscr{D}}, 0) \neq 0$.

In the second case we use the Principle of Nonzero Degree [8, Chapter 2, Corollary 4.1] to conclude that there exists a sequence $\{u_n''\} \subset \mathscr{D}$ such that

$$Au_n'' - \tilde{C}(\lambda_n'', u_n'') = 0.$$

Thus, in both cases, we have sequences $\{\lambda_n\}$, $\{u_n\}$ such that $\lambda_n \to \infty$, $\{u_n\} \subset \overline{\mathscr{D}}$ and

$$Au_n - \tilde{C}(\lambda_n, u_n) = 0.$$

However, this provides us with the desired contradiction because the sequence $\{Au_n\}$ is bounded, by our assumption on the boundedness of the operator A, while the sequence $\{\tilde{C}(\lambda_n, u_n)\}$ is unbounded by virtue of property (7) of the operator $\tilde{C}$. It follows that there exists $\tilde{\lambda}$ such that (23) holds.

Now, we consider the family of operators

$$A_t u \equiv Au - \tilde{C}(\tilde{\lambda} t, u), \qquad u \in \overline{\mathscr{D}}, \quad t \in [0, 1],$$

which satisfies condition $\alpha_o^{(t)}(\partial \mathscr{D})$ in [8]. It is possible that there exists a number $\tilde{t} \in (0, 1)$ and $\tilde{u} \in \partial \mathscr{D}$ such that

$$A\tilde{u} - \tilde{C}(\tilde{\lambda}\tilde{t}, \tilde{u}) = 0. \tag{24}$$

In the opposite case, the family A_t is a homotopy of mappings satisfying condition $(S)_+$. By [2, Chapter 2, Theorem 4.1], we have

$$Deg(A_0, \overline{\mathscr{D}}, 0) = Deg(A_1, \overline{\mathscr{D}}, 0).$$

This is, however, impossible because the left-hand side is not zero, by our assumption on A and Remark 3, while the right-hand side equals zero by (23). Consequently, we have shown the existence of $\tilde{\lambda}$, $\tilde{u}$, $\tilde{t}$ satisfying (24). From (24) we obtain (22) with $u_0 = \tilde{u}$, $\lambda_0 = \tilde{\lambda}\tilde{t}$. The proof is complete. ∎

In the following theorems we assume that the operator A is densely defined. We formulate results where A is either m-accretive or maximal monotone.

Definition 2. We say that the operator $A: X_1 \supset D(A) \to X_2$ is of proper type on the set $F \subset D(A)$ if for every sequence $\{u_n\} \subset F$ such that $u_n \rightharpoonup u_0 \in X_1$ and $\{Au_n\}$ converges strongly we have $u_n \to u_0$.

Definition 3. We say that the operator $A: X_1 \supset D(A) \to X_2$ satisfies condition (A_∞) on the bounded set $F \subset D(A)$ if there is no sequence $\{u_n\} \subset F$ such that

$$\|Au_n\|_2 \to \infty \qquad \text{and} \qquad \|Au_n\|_2^{-1} Au_n \to h,$$

for an element $h \in X_2$.

THEOREM 2. *Let X be a real reflexive and locally uniformly convex space with locally uniformly dual space X^*. Assume that $A: X \supset D(A) \to X^*$ is maximal monotone, of proper type on $D(A) \cap \partial\mathscr{D}$, and satisfying condition* (A_∞) *on $D(A) \cap \overline{\mathscr{D}}$. Assume that $\overline{\mathscr{D}} \subset \overline{D(A)}$, $0 \in D(A) \cap \mathscr{D}$, $Au \neq 0$ for $u \in D(A) \cap \partial\mathscr{D}$ and $A0 = 0$. Let $C: \mathscr{R}_+ \times \partial\mathscr{D} \to X^*$ be a compact operator satisfying condition* (ii) *of Lemma* 1 *(with $X = X^*$) and $C(0, u) = 0$, for $u \in \partial\mathscr{D}$. Assume that the closure of the set E defined by* (4) *is compact. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that*

$$Au_0 - C(\lambda_0, u_0) = 0. \tag{25}$$

Proof. Under the conditions of the theorem the operator $(\varepsilon J + A)^{-1}: X^* \to D(A)$ is well-defined and bounded for every $\varepsilon > 0$.

Given $\varepsilon > 0$, we shall first find $(\lambda_\varepsilon, u_\varepsilon) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that

$$\varepsilon J u_\varepsilon + A u_e - C(\lambda_\varepsilon, u_\varepsilon) = 0. \tag{26}$$

In order to achieve this, we shall prove the existence of some $\tilde{\lambda} \in \mathscr{R}_+$ such that

$$deg(I - (\varepsilon J + A)^{-1} \tilde{C}(\lambda, \cdot), \overline{\mathscr{D}}, 0) = 0, \qquad \text{for all} \quad \lambda \geqslant \tilde{\lambda}, \tag{27}$$

where *deg* stands for the Leray–Schauder degree and the operator $\tilde{C}(\lambda, \cdot)$ is constructed according to Lemma 1 and is compact by virtue of Remark 1. If (27) is false, then, as in the proof of (23), we can find sequences $\{\lambda_n\} \subset \mathscr{R}_+$, $\{u_n\} \subset \overline{\mathscr{D}}$ such that

$$\varepsilon J u_n + A u_n - \tilde{C}(\lambda_n, u_n) = 0 \tag{28}$$

and $\lambda_n \to \infty$, for $n \to \infty$. From (7), it follows that $\|\tilde{C}(\lambda_n, u_n)\|_* \to \infty$ as $n \to \infty$. Using Remark 3, we obtain that some subsequence of the sequence $\{\|Au_n\|_*^{-1} Au_n\}$ is strongly convergent. Since $\|Au_n\|_* \to \infty$, we have the contradiction of condition (A_∞).

We now consider the family of operators

$$I - (\varepsilon J + A)^{-1} \tilde{C}(t\tilde{\lambda}, \cdot): \overline{\mathscr{D}} \to X, \qquad 0 \leqslant t \leqslant 1,$$

which are well-defined compact displacements of the identity. From the equalities (27),

$$deg(I - (\varepsilon J + A)^{-1} \tilde{C}(0, \cdot), \overline{\mathscr{D}}, 0) = deg(I, \overline{\mathscr{D}}, 0) = 1$$

and the homotopy invariance property of the degree it follows that there exist $\tilde{t} \in (0, 1)$ and $\tilde{u} \in \partial\mathscr{D}$ such that

$$\varepsilon J\tilde{u} - A\tilde{u} - C(\tilde{t}\tilde{\lambda}, \tilde{u}) = 0.$$

We have also used the property (6) of the operator $\tilde{C}$. Taking $u_\varepsilon = \tilde{u}$, $\lambda_\varepsilon = \tilde{t}\tilde{\lambda}$, we find the desired λ_ε, u_ε satisfying the equation (26).

Let $\{\varepsilon_n\}$ be a positive sequence tending to zero and let $\bar{u}_n = u_{\varepsilon_n}$, $\bar{\lambda}_n = \lambda_{\varepsilon_n}$. Passing to subsequences if necessary, we may assume that $\bar{u}_n \rightharpoonup u_0$. For the sequence $\{\bar{\lambda}_n\}$ we have two possibilities:

$$\text{(i)} \quad \bar{\lambda}_n \to \lambda_0 \in \overline{\mathscr{R}_+} \qquad \text{and} \qquad \text{(ii)} \quad \bar{\lambda}_n \to \infty.$$

If $\bar{\lambda}_n \to \lambda_0$, then we may also assume that $C(\bar{\lambda}_n, \bar{u}_n) \to h$. From (26), with $\varepsilon = \varepsilon_n$, we obtain $A\bar{u}_n \to h$, which, in view of the properness of A on $D(A) \cap \partial\mathscr{D}$, implies the strong convergence of $\bar{u}_n$ to u_0. Passing to the limit in the equality

$$\varepsilon_n J\bar{u}_n + A\bar{u}_n - C(\bar{\lambda}_n, \bar{u}_n) = 0,$$

we conclude that u_0, λ_0 satisfy the equation (25). Since $Au_0 \neq 0$ for $u_0 \in D(A) \cap \partial\mathscr{D}$, we must have $\lambda_0 > 0$.

In the case (ii), the equality (26), (with $\varepsilon = \varepsilon_n$, $\bar{u}_n = u_{\varepsilon_n}$ and $\bar{\lambda}_n = \bar{\lambda}_{\varepsilon_n}$), can be treated as the equality (28). By repeating the argument about (28), we can see that the case (ii) is not possible. This completes the proof. ∎

DEFINITION 4. We say that the operator $A: X_1 \supset D(A) \to X_2$ satisfies condition $(A_\infty^{(0)})$ on the set $F \subset D(A)$ if for an arbitrary sequence $\{u_n\} \subset F$, such that $\|Au_n\|_2 \to \infty$, the sequence $\{\|Au_n\|_2^{-1} Au_n\}$ converges weakly to zero.

Remark 4. From the proofs of Theorem 2 and Lemma 2 is follows that the assertion of Theorem 2 is valid with the following changes in the conditions on the operators A, C: instead of the condition (A_∞), the operator A satisfies condition $(A_\infty^{(0)})$; instead of the assumption that the closure of the set E is compact, the operator C is such that the weak closure of the set E does not contain zero.

We now formulate a corresponding result for m-accretive operators.

THEOREM 3. *Let X be a real Banach space and $\mathscr{D}$ a bound open subset of X. Assume that $A: X \supset D(A) \to X$ is m-accretive and of proper type on $D(A) \cap \partial\mathscr{D}$. Assume that A satisfies condition (A_∞) on $D(A) \cap \overline{\mathscr{D}}$ and that $\overline{\mathscr{D}} \subset \overline{D(A)}$, $0 \in D(A) \cap \mathscr{D}$, $A(0) = 0$ and $Au \neq 0$ for $u \in D(A) \cap \partial\mathscr{D}$. Let $C: \overline{\mathscr{R}_+} \times \overline{\mathscr{D}} \to X$ be a compact operator with $C(0, u) \equiv 0$ for $u \in \partial\mathscr{D}$ and satisfying condition* (ii) *of Lemma* 1 *(with $X_2 = X$). Assume that the closure of the set E defined by* (4) *is compact. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that*

$$Au_0 - C(\lambda_0, u_0) = 0.$$

Proof. By the conditions of the theorem, for every $\varepsilon > 0$ the operator $(\varepsilon I + A)^{-1}: X \to D(A)$ is well-defined, continuous and bounded. Replacing in the proof of Theorem 2 the operator $(\varepsilon J + A)^{-1}$ by the operator $(\varepsilon I + A)^{-1}$ and repeating our arguments there, we obtain the proof of Theorem 3. ▮

It is possible to prove analogues of Theorems 2 and 3 for cases where the operators C are continuous and bounded while the operators $(\varepsilon J + A)^{-1}$ are compact.

We shall consider operators $C: \overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to X_2$, where $\mathscr{D}$ is a bounded open set in X_1, and assume the following condition which relates the operators A and C.

($\mathscr{C}_1$) There exists a positive number $\mathscr{N}$ such that the weak closure of the set

$$G = \left\{ \frac{C(\lambda, u)}{\|C(\lambda, u)\|_2} : \lambda \geqslant \mathscr{N}, u \in \bar{\mathscr{D}}, \|Ju + Au\|_2 \leqslant 2M(\lambda) \right\}$$

does not contain zero, where

$$M(\lambda) = \sup\{ \|C(\lambda, u)\|_2 : u \in \bar{\mathscr{D}} \};$$

($\mathscr{C}_2$) we have

$$\lim_{\lambda \to \infty} \bar{m}(\lambda) = +\infty, \qquad \text{where} \quad \bar{m}(\lambda) = \inf\{ \|C(\lambda, u)\|_2 : u \in \bar{\mathscr{D}} \}.$$

THEOREM 4. *Let X be a real, locally uniformly convex Banach space with locally uniformly convex dual space X^*. Let $\mathscr{D}$ be an open bounded subset of X. Assume that $A: X \supset D(A) \to X^*$ is maximal monotone with $(A + J)^{-1}$ compact. Assume that A is of proper type on $D(A) \cap \partial\mathscr{D}$ and satisfies condition $(A_\infty^{(0)})$ on $D(A) \cap \bar{\mathscr{D}}$. Assume that $\bar{\mathscr{D}} \subset \overline{D(A)}$, $0 \in D(A) \cap \mathscr{D}$, $A(0) = 0$ and $Au \neq 0$ for $u \in D(A) \cap \partial\mathscr{D}$. Let $C: \overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to X^*$ be a bounded continuous operator such that $C(0, u) \equiv 0$ for $u \in \partial\mathscr{D}$ and satisfying conditions ($\mathscr{C}_1$) and ($\mathscr{C}_2$) with $X_1 = X$ and $X_2 = X^*$. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that*

$$Au_0 - C(\lambda_0, u_0) = 0.$$

Proof. By the conditions of the theorem, the operator $(\varepsilon J + A)^{-1}: X^* \to D(A)$ is well-defined and compact for $\varepsilon > 0$. We choose a sufficiently small $\varepsilon > 0$ and prove the existence of $\tilde{\lambda} \in \mathscr{R}_+$ such that

$$deg(I - (\varepsilon J + A)^{-1} C(\lambda, \cdot), \bar{\mathscr{D}}, 0) = 0, \qquad \text{for} \quad \lambda \geqslant \tilde{\lambda}, \tag{29}$$

where *deg* denotes the Leray–Schauder degree. If (29) is not true, then, as in the proof of (23), we can find sequences $\{\lambda_n\} \subset \mathscr{R}_+$, $\{u_n\} \subset \bar{\mathscr{D}}$ such that

$$\varepsilon J u_n + A u_n - C(\lambda_n, u_n) = 0 \tag{30}$$

and $\lambda_n \to \infty$ for $n \to \infty$. For sequences $\{u_n\}$ satisfying the equality (30), we have the estimate

$$\|Ju_n + Au_n\|_* \leqslant \|C(\lambda_n, u_n)\|_* + \|Ju_n\|_* \leqslant 2M(\lambda_n)$$

for all large *n*. Using the property $(\mathscr{C}_1)$, we may assume that the sequence $\{\|C(\lambda_n, u_n)\|_*^{-1} \cdot C(\lambda_n, u_n)\}$ converges weakly to some element h_0 such that $h_0 \neq 0$. Then from (30) we obtain

$$\|Au_n\|_*^{-1} Au_n \rightharpoonup h_0.$$

Thus, we have reached a contradiction to condition $(A_\infty^{(0)})$ because $\|Au_n\|_* \to \infty$. Therefore, the equality (29) is proved. The end of the proof of this theorem, using the equality (29), follows the steps of the corresponding part of the proof of Theorem 2. It is therefore omitted. ∎

We now formulate the analog of Theorem 4 for *m*-accretive operators.

THEOREM 5. *Let X be a real reflexive Banach space and let $\mathscr{D}$ be an open bounded set in X. Assume that $A\colon X \supset D(A) \to X$ is m-accretive, of proper type on $D(A) \cap \partial\mathscr{D}$ and such that the resolvent $(A+I)^{-1}$ is compact. Assume, further, that the operator A satisfies condition $(A_\infty^{(0)})$ on $D(A) \cap \bar{\mathscr{D}}$, $\bar{\mathscr{D}} \subset \overline{D(A)}$, $0 \in D(A) \cap \mathscr{D}$, $A(0) = 0$ and $Au \neq 0$ for $u \in D(A) \cap \partial D$. Let $C\colon \overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to X$ be a bounded continuous operator which satisfies conditions $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ with $X_1 = X_2 = X$. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that*

$$Au_0 - C(\lambda_0, u_0) = 0.$$

The proof of Theorem 5 is analogous to the proof of Theorem 4. It is therefore omitted.

Remark 5. If in the condition of Theorems 2 and 3 the operator A is α-homogeneous, then the assumption that A satisfies condition (A_∞) follows from other appropriate assumptions on it. For example, we note here that a homogeneous *m*-accretive operator A satisfies the condition (A_∞) on an arbitrary bounded set $F \subset D(A)$ if $0 \in D(A)$ and $A(0) = 0$. In fact, let $\{u_n\} \subset F$ be an arbitrary sequence with $\|Au_n\| \to \infty$ and $\|Au_n\|^{-1} Au_n \to h$, where h is some element of X with $\|h\| = 1$. Then for the sequence

$$v_n = \|Au_n\|^{-1/\alpha} u_n$$

we have $v_n \to 0$ and $Av_n \to h$. Since A is closed, we must have $h = A(0) = 0$. This is a contradiction to $\|h\| = 1$.

4. EIGENVECTORS FOR NONLINEAR ELLIPTIC PROBLEMS

We now provide various applications of the theoretical results of the preceding section involving the existence of eigenvectors of nonlinear elliptic operators.

Problem 1. We start with the eigenvalue problem for nondivergent nonlinear second order operators. Let Ω be a bounded open set in $\mathscr{R}^n$ with boundary $\partial\Omega$ of class $\mathscr{C}^2$. Assume that the functions $a_{ij}(x, u, p)$, $i, j = 1, ..., n$, satisfy the following conditions:

($\mathscr{A}_1^{(1)}$) The functions $a_{ij}(x, u, p)$ are defined and continuous for $x \in \bar{\Omega}$, $u \in \mathscr{R}$ and $p \in \mathscr{R}^n$;

($\mathscr{A}_2^{(1)}$) there exists a positive nondecreasing function $\nu: \overline{\mathscr{R}_+} \to \mathscr{R}_+$ such that

$$\sum_{i, j=1}^{n} a_{ij}(x, u, p)\, \xi_i \xi_j \geqslant \nu(|u| + |p|) \sum_{i=1}^{n} \xi_i^2 \tag{31}$$

for $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathscr{R}^n$, $x \in \bar{\Omega}$, $u \in \mathscr{R}$, $p \in \mathscr{R}^n$.

Assume that the function $C(\lambda, x, u, p)$ satisfies the following conditions:

($\mathscr{C}_1^{(1)}$) $C(\lambda, x, u, p)$ is defined and continuous for $(\lambda, x, u, p) \in \overline{\mathscr{R}_+} \times \bar{\Omega} \times \mathscr{R} \times \mathscr{R}^n$ and $C(0, x, u, p) = 0$ for all $(x, u, p) \in \bar{\Omega} \times \mathscr{R} \times \mathscr{R}^n$;

$\mathscr{C}_2^{(1)}$) there exists a function $f(\lambda)$ and a continuous function $C'(x, u, p)$, defined for $(x, u, p) \in \bar{\Omega} \times \mathscr{R} \times \mathscr{R}^n$, such that $f(\lambda) \geqslant 1$ and

$$\lim_{\lambda \to \infty} f(\lambda) = \infty, \qquad \lim_{\lambda \to \infty} \frac{1}{f(\lambda)}\, C(\lambda, x, u, p) = C'(x, u, p), \tag{32}$$

where the second limit is uniform w.r.t. x, u, p on an arbitrary bounded set $G \subset \bar{\Omega} \times \mathscr{R} \times \mathscr{R}^n$;

($C_3^{(1)}$) there exists a nonnegative continuous function $C''(x, r)$, defined for $x \in \bar{\Omega}$, $r \in \overline{\mathscr{R}_+}$, such that: for an arbitrary number $R \in \mathscr{R}$, we have

$$|C'(x, u, p)| \geqslant C''(x, R), \qquad \int_\Omega C''(x, R)\, dx > 0 \tag{33}$$

for all $x \in \bar{\Omega}$ and all $(u, p) \in \mathscr{R} \times \mathscr{R}^n$ with $|u| + |p| \leqslant R$.

THEOREM 6. *Assume that $\partial\Omega$ belongs to the class $\mathscr{C}^2$ and that the functions $a_{ij}(x, u, p)$, $C(\lambda, x, u, p)$ satisfy the conditions ($\mathscr{A}_1^{(1)}$), ($\mathscr{A}_2^{(1)}$) and ($\mathscr{C}_1^{(1)}$)–($\mathscr{C}_3^{(1)}$), respectively. Let q be a number with $q > n$. Then for an* *arbitrary bounded open set* $\mathscr{D} \subset X^{(1)} = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, *such that* $0 \in \mathscr{D}$, *there exist* $\lambda_0 \in \mathscr{R}_+$, $u_0 \in \partial\mathscr{D}$ *satisfying the eigenvalue problem*

$$\sum_{i,j=1}^{n} a_{ij}\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right)\frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} - C\left(\lambda_0, x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) = 0, \qquad x \in \Omega,$$
$$u_0(x) = 0, \qquad x \in \partial\Omega. \tag{34}$$

Proof. We shall reduce the eigenvalue problem (34) to an eigenvalue problem for abstract operators satisfying the conditions of Theorem 1 in the space $X^{(1)} = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. By our assumptions on Ω and q, we have the compact embedding $W^{2,q}(\Omega) \subset \mathscr{C}^{1,\delta}(\bar{\Omega})$, for some $\delta > 0$. Consequently, all the functions $u \in \bar{\mathscr{D}}$ satisfy the inequality

$$\|u\|_{C^{1,\delta}(\Omega)} \leqslant K_0, \tag{35}$$

for some positive constant K_0.

We define, for $u \in \bar{\mathscr{D}}$, the linear elliptic operator $L(u)$: $X^{(1)} \to L^q(\Omega)$ by the equality

$$L(u)\, v(x) = \sum_{i,j=1}^{n} a_{ij}\left(x, u(x), \frac{\partial u(x)}{\partial x}\right)\frac{\partial^2 v(x)}{\partial x_i \partial x_j}.$$

We shall denote the norms in the Lebesgue space $L^p(\Omega)$ and the Sobolev space $W^{k,p}(\Omega)$ by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively. From well-known a priori estimates (see [1]), we have the inequality

$$\|v\|_{2,q} \leqslant C_1 \, \|L(u)\, v(\cdot)\|_q \leqslant C_2 \, \|v\|_{2,q} \tag{36}$$

for all functions $u \in \bar{\mathscr{D}}$, $v \in X^{(1)}$ with constants C_1, C_2 independent of u, v.

We introduce the operators A: $\bar{\mathscr{D}} \to [X^{(1)}]^*$, C: $\overline{\mathscr{R}_+} \times \bar{\mathscr{D}} \to [X^{(1)}]^*$ by the equalities

$$\langle Au, \phi \rangle = \int_\Omega \psi_q(L(u)\, u(x)) \cdot L(u)\, \phi(x)\, dx,$$
$$\langle C(\lambda, u), \phi \rangle = \int_\Omega \psi_q\left(C\left(\lambda, x, u(x), \frac{\partial u(x)}{\partial x}\right)\right) L(u)\, \phi(x)\, dx, \tag{37}$$

where $\psi_q(s) = |s|^{q-2} \cdot s$, $s \geqslant 0$. It is a simple matter to verify that these operators are well-defined and continuous.

The operator A, defined by (37), is of type $(S)_+$. Recall that if $\{u_n(x)\}$ is such that $u_n \in \bar{\mathscr{D}}$ and $u_n \rightharpoonup u$ in $X^{(1)}$, then from condition $(\mathscr{A}_1^{(1)})$ we have

$$\lim_{n\to\infty} \{ \sup_{\|\phi\|_{2,q}=1} \|L(u_n)\, \phi(\cdot) - L(u_0)\, \phi(\cdot)\|_q \} = 0. \tag{38}$$

If the sequence $\{u_n(x)\}$ is such that

$$\limsup_{n\to\infty} \langle Au_n, u_n - u_0 \rangle \leqslant 0,$$

then from (38) we obtain

$$\limsup_{n\to\infty} \int_\Omega \psi_q(L(u_0)\, u_n(x)) \cdot L(u_0)[u_n(x) - u_0(x)]\, dx \leqslant 0. \tag{39}$$

From the fact that $u_n \rightharpoonup u_0$, it follows that

$$\lim_{n\to\infty} \int_\Omega \psi_q(L(u_0)\, u_0(x)) \cdot L(u_0)[u_n(x) - u_0(x)]\, dx = 0. \tag{40}$$

Taking into consideration the trivial inequality

$$[\psi_q(s_1) - \psi_q(s_2)](s_1 - s_2) \geqslant C_q\, |s_1 - s_2|^q, \qquad s_1, s_2 \in \mathscr{R}, \tag{41}$$

for some positive constant C_q, we obtain from (39), (40) the inequality

$$\limsup_{n\to\infty} \int_\Omega |L(u_0)[u_n(x) - u_0(x)]|^q\, dx \leqslant 0$$

and the strong convergence of u_n to u follows from the a priori estimate (36).

The operator A, defined by (37), satisfies the condition

$$\langle Au, u \rangle > 0, \qquad u \in \bar{\mathscr{D}}, \quad u \neq 0.$$

Consequently, the degree $Deg(A, \bar{\mathscr{D}}, 0)$ is well-defined, and, by [8, Chapter 2, Theorem 4.4], this degree equals one.

The operator C, defined by (37), is compact by virtue of the compactness of the embedding of $W^{2,q}(\bar{\Omega})$ in $\mathscr{C}^{1,\delta}(\Omega)$. We are going to verify that this operator satisfies also the conditions (i), (ii) of Lemma 1. From (36) and the definition of the operator $C(\lambda, u)$, it follows that the inequalities

$$\begin{aligned} C_3 \left\| C\left(\lambda, \cdot, u(\cdot), \frac{\partial u(\cdot)}{\partial x}\right) \right\|_q^{q-1} &\leqslant \| C(\lambda, u(\cdot)) \|_*^{(1)} \\ &\leqslant C_4 \left\| C\left(\lambda, \cdot, u(\cdot), \frac{\partial u(\cdot)}{\partial x}\right) \right\|_q^{q-1} \end{aligned} \tag{42}$$

hold for all $u \in \overline{\mathscr{D}}$, where C_3, C_4 are some positive constants independent of u. In (42), and later, $\|\cdot\|_*^{(1)}$ is a norm on $[X^{(1)}]^*$.

Let ε be an arbitrary positive number. Using conditions $(\mathscr{C}_2^{(1)})$, $(\mathscr{C}_3^{(1)})$ and the inequality (35), it is possible to find Λ_ε such that for $\lambda \geqslant \Lambda_\varepsilon$, $u \in \overline{\mathscr{D}}$, we have the estimate

$$\left| C(\lambda, x, u(x), \frac{\partial u(x)}{\partial x}) \right| \geqslant f(\lambda)[C''(x, K_0) - \varepsilon]. \tag{43}$$

From this inequality and (32), (33), we obtain that the operator $C(\lambda, u)$, defined by (37), satisfies condition (ii) of Lemma 1.

We now verify that the closure of the set

$$E^{(1)} = \left\{ \frac{C(\lambda, u)}{\|C(\lambda, u)\|_*^{(1)}} : u \in \overline{\mathscr{D}}, \lambda \in \mathscr{R}_+, \|C(\lambda, u)\|_*^{(1)} \geqslant 1 \right\}$$

is compact in order to guarantee that the operator C satisfies the condition (i) of Lemma 1. Given a sequence

$$\{[\|C(\lambda_n, u_n)\|_*^{(1)}]^{-1} C(\lambda_n, u_n)\},$$

with $u_n \in \overline{\mathscr{D}}$, $\lambda_n \in \mathscr{R}_+$ and $\|C(\lambda_n, u_n)\|_*^{(1)} \geqslant 1$, we shall show the existence of a strongly convergent subsequence. Taking into consideration the compactness of the operator C, it is necessary to consider only the case $\lambda_n \to \infty$. From (32), (33), (42), and (43), we have, for sufficiently large n, the inequality

$$C_5[f(\lambda_n)]^{q-1} \leqslant \|C(\lambda_n, u_n)\|_*^{(1)} \leqslant C_6[f(\lambda_n)]^{q-1}, \tag{44}$$

with some positive constants C_5, C_6 which are independent of λ_n, u_n. Passing to a subsequence, if necessary, we may assume that $u_n(x)$ converges to $u_0(x)$ in $\mathscr{C}^1(\overline{\Omega})$. By condition $(\mathscr{C}_2^{(1)})$, we obtain that the convergence

$$\frac{1}{f(\lambda_n)} C\left(\lambda_n, x, u_n(x), \frac{\partial u_n(x)}{\partial x}\right) \to C'\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right)$$

is uniform on $\overline{\Omega}$. Taking into consideration (44), we establish the compactness of the set $\overline{E^{(1)}}$.

Thus, all the assumptions of Theorem 1 are satisfied. Consequently, there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times \partial\mathscr{D}$ such that the equality

$$\int_\Omega \left\{\psi_q(L(u_0)\, u_0(x)) - \psi_q\left(C\left(\lambda_0, u_0(x), \frac{\partial u_0(x)}{\partial x}\right)\right)\right\} L(u_0)\, \phi(x)\, dx = 0 \tag{45}$$

is valid for an arbitrary function $\phi \in X^{(1)}$. From the existence result for linear elliptic operators in [1], it follows that it is possible to choose a function $\phi_0 \in X^{(1)}$ so that

$$L(u_0)\,\phi_0(x) = L(u_0)\,u_0(x) - C\left(\lambda_0, x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right).$$

Substituting this function $\phi_0(x)$ in (45), we obtain from (41)

$$L(u_0)\,u_0(x) - C\left(\lambda_0, x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) = 0.$$

Therefore, $\lambda_0, u_0(x)$ satisfy the eigenvalue problem (34) and the proof of Theorem 6 is complete. ▮

Problem 2. In the applications that follow we shall consider densely defined and unbounded operators A. We denote by $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ a multi-index with nonnegative integer coordinates. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$\mathscr{D}^{\alpha} u(x) = \left(\frac{\partial}{\partial x_n}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u(x), \qquad \mathscr{D}^m u(x) = \{\mathscr{D}^{\alpha} u(x) : |\alpha| = m\}.$$

Let Ω be a bounded open set in $\mathscr{R}^n$ with boundary $\partial\Omega$ of class $\mathscr{C}^{m+1}$. We assume that the functions $A_\alpha(x, \xi^{(m)})$, $|\alpha| = m$, $\xi^{(m)} = \{\xi_\alpha : |\alpha| = m\}$ satisfy the following conditions:

($\mathscr{A}_1^{(2)}$) The functions $A_\alpha(x, \xi^{(m)})$ are defined and differentiable w.r.t. all their arguments for $x \in \bar{\Omega}$, $\xi^{(m)} \in \mathscr{R}^{N(m)}$, where $N(m)$ is the number of different multi-indices α such that $|\alpha| = m$. In addition, $A_\alpha(x, 0) \equiv 0$ for $|\alpha| = m$, $x \in \bar{\Omega}$;

($\mathscr{A}_2^{(2)}$) there exist positive constants K_1, K_2 such that the inequalities

$$\sum_{|\alpha| = |\beta| = m} A_{\alpha\beta}(x, \xi^{(m)})\,\eta_\alpha \eta_\beta \geqslant K_1 \sum_{|\alpha| = m} \eta_\alpha^2,$$

$$\sum_{|\alpha| = |\beta| = m} |A_{\alpha\beta}(x, \xi^{(m)})| \leqslant K_2, \qquad \sum_{|\alpha| = m} \sum_{i=1}^{n} |A_{\alpha,i}(x, \xi^{(m)})| \leqslant K_2(1 + |\xi^{(m)}|) \tag{46}$$

are satisfied for $x \in \bar{\Omega}$, $\xi^{(m)} \in \mathscr{R}^{N(m)}$, $\eta = \{\eta_\alpha : |\alpha| = m\} \in \mathscr{R}^{N(m)}$. Here,

$$A_{\alpha\beta}(x, \xi^{(m)}) = \frac{\partial}{\partial \xi_\beta} A_\alpha(\alpha, \xi^{(m)}), \qquad A_{\alpha,i}(x, \xi^{(m)}) = \frac{\partial}{\partial x_i} A_\alpha(x, \xi^{(m)}).$$

We also assume that the functions $C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)})$, $|\gamma| \leqslant m-1$, $\xi^{(j)} = \{\xi_\alpha : |\alpha| = j\}$, $j=0, 1, ..., m-1$, satisfy the following conditions:

($\mathscr{C}_1^{(2)}$) The functions $C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)})$ are defined and continuous for $(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)}) \in \overline{\mathscr{R}_+} \times \bar{\Omega} \times \mathscr{R} \times \cdots \times \mathscr{R}^{N(m-1)}$. Moreover, $C_\gamma(0, x, \xi^{(0)}, ..., \xi^{(m-1)}) = 0$;

($\mathscr{C}_2^{(2)}$) there exists a function $f(\lambda)$ and a continuous function $C'_\gamma(x, \xi^{(0)}, ..., \xi^{(m)})$, $|\gamma| \leqslant m-1$ such that $f(\lambda) \geqslant 1$ and

$$|C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)})| \leqslant f(\lambda)\left[1 + \sum_{j=0}^{m-1} |\xi^{(j)}|\right],$$

$$\lim_{\lambda\to\infty} f(\lambda) = \infty, \qquad \lim_{\lambda\to\infty} \frac{1}{f(\lambda)}\, C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)}) = C'_\gamma(x, \xi^{(0)}, ..., \xi^{(m-1)}) \tag{47}$$

for all $x \in \Omega$, $\xi^{(j)} \in \mathscr{R}^{N(j)}$, $j=0, ..., m-1$, where the second limit is uniform w.r.t. $x, \xi^{(0)}, ..., \xi^{(m-1)}$ on an arbitrary bounded set;

($\mathscr{C}_3^{(2)}$) there exists a function $\omega \in \mathscr{C}_0^\infty(\Omega)$ and a continuous functions $C''(x)$, $x \in \bar{\Omega}$, such that the inequalities

$$\sum_{|\gamma| \leqslant m-1} C'_\gamma(x, \xi^{(0)}, ..., \xi^{(m-1)})\, \mathscr{D}^\gamma \omega(x) \geqslant C''(x)$$
$$\int_\Omega C''(x)\, dx > 0 \tag{48}$$

are valid for $x \in \bar{\Omega}$, $\xi^{(j)} \in \mathscr{R}^{N(j)}$, $j=0, 1, ..., m-1$.

THEOREM 7. *Assume that $\partial\Omega$ is of class $\mathscr{C}^{m+1}$ and that the functions $A_\alpha(x, \xi^{(m)})$, $C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)})$, $|\alpha| = m$, $|\gamma| \leqslant m-1$, satisfy the conditions ($\mathscr{A}_1^{(2)}$), ($\mathscr{A}_2^{(2)}$) and ($\mathscr{C}_1^{(2)}$)–($\mathscr{C}_3^{(2)}$), respectively. Then, for an arbitrary bounded open set $\mathscr{D} \subset W_0^{m-1,2}(\Omega)$ such that $0 \in \mathscr{D}$, there exist $\lambda_0 \in \mathscr{R}_+$ and $u_0 \in \partial\mathscr{D}$ satisfying the eigenvalue problem*

$$\sum_{|\alpha|=m} (-1)^m\, \mathscr{D}^\alpha A_\alpha(x, \mathscr{D}^m u_0)$$
$$- \sum_{|\gamma| \leqslant m-1} (-1)^{|\gamma|}\, \mathscr{D}^\gamma C_\gamma(\lambda_0, x, u_0, ..., \mathscr{D}^{m-1} u_0) = 0,$$
$$u_0 \in W_0^{m,2}(\Omega) \cap W^{m+1,2}(\Omega). \tag{49}$$

Before we give the proof of Theorem 7, we are going to reduce the problem to an abstract problem where the operators satisfy the conditions of Theorem 4 for the space $X^{(2)} = W_0^{m-1,2}(\Omega)$. At first, we study the properties of the operator A which represents the principal part of the differential equation.

We define the operator $A: X^{(2)} \supset D(A) \to [X^{(2)}]^*$, with $D(A) = \{u \in X^{(2)} : u \in W_0^{m,2}(\Omega) \cap W^{m+1,2}(\Omega)\}$, by

$$\langle Au, \phi \rangle = -\sum_{|\alpha|=m} \int_\Omega \mathscr{D}^{\alpha'} A_\alpha(x, \mathscr{D}^m u)\, \mathscr{D}^{\alpha-\alpha'}\phi(x)\, dx, \tag{50}$$

where $\alpha' = (\alpha'_1, ..., \alpha'_n)$ is a multi-index of length one which is uniquely defined by $\alpha = (\alpha_1, ..., \alpha_n)$ as follows: $\alpha'_{j_0} = 1$ if $\alpha_j = 0$ for $j < j_0$, $\alpha_{j_0} \neq 0$. The desired properties of the operator A are given in the following lemma.

LEMMA 3. *The operator A, defined by* (5), *is a monotone operator acting from $D(A)$ to $[X^{(2)}]^*$. Moreover, the inverse operator $A^{-1}: [X^{(2)}]^* \to D(A)$ exists and is compact.*

Proof. It is a simple matter to verify that the operator A is well-defined. Its monotonicity property follows from the inequality

$$\begin{aligned}
&\langle Au - Av, u - v \rangle \\
&\quad = \sum_{|\alpha|=m} \int_\Omega [A_\alpha(x, \mathscr{D}^m u) - A_\alpha(x, \mathscr{D}^m v]\, \mathscr{D}^\alpha(u-v)\, dx \\
&\quad = \int_0^1 \left\{ \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x, t\mathscr{D}^m u + (1-t)\, \mathscr{D}^m v)\, \mathscr{D}^\alpha(u-v)\, \mathscr{D}^\beta(u-v) \right\} dx\, dt \\
&\quad \geqslant C_7\, \|u-v\|^2_{m,2},
\end{aligned} \tag{51}$$

which is valid for arbitrary functions $u, v \in \mathscr{D}(A)$. Here, $\|\cdot\|_{m,2}$ is a norm on $W^{m,2}(\Omega)$.

Let h be an arbitrary functional from $[X^{(2)}]^*$. We shall find $u_h \in D(A)$ such that

$$Au_h = h. \tag{52}$$

We can choose functions $h_\gamma \in L^2(\Omega)$, $|\gamma| \leqslant m-1$, such that

$$\langle h, \phi \rangle = \sum_{|\gamma| \leqslant m-1} \int_\Omega h_\gamma(x)\, \mathscr{D}^\gamma \phi(x)\, dx, \qquad \sum_{|\gamma| \leqslant m-1} \|h_\gamma\|_2 \leqslant C_8\, \|h\|_*^{(2)}, \tag{53}$$

where $\|\cdot\|_2$, $\|\cdot\|_*^{(2)}$ are norms on $L^2(\Omega)$ and $[X^{(2)}]^*$, respectively, and C_8 is a constant independent of h.

Consider the boundary value problem

$$\sum_{|\alpha|=m} (-1)^{|\alpha|}\, \mathscr{D}^\alpha A_\alpha(x, \mathscr{D}^m u) = \sum_{|\gamma| \leqslant m-1} (-1)^{|\gamma|}\, \mathscr{D}^\gamma h_\gamma(x) \tag{54}$$

in the space $W_0^{m,2}(\Omega)$. This problem can be reduced to an operator equation with a coercive monotone operator by using a standard argument. Thus, there exists a solution $u_h \in W_0^{m,2}(\Omega)$ of the equation (54). We shall prove that $u_h \in W^{m+1,2}(\Omega)$.

Define, for $|\gamma| \leqslant m-1$, a function $v_\gamma \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ as a solution of the equation

$$\Delta v_\gamma x = h_\gamma(x), \qquad x \in \Omega,$$

where Δ is the Laplacian operator. It is well-known that the estimate

$$\|v_\gamma\|_{2,2} \leqslant C_9 \, \|h_\gamma\|_2$$

holds with constant C_9 independent of h_γ. Using functions $v_\gamma(x)$, we can rewrite the problem (54) in the form

$$\sum_{|\alpha|=m} (-1)^{|\alpha|} \, \mathscr{D}^\alpha A_\alpha(x, \mathscr{D}^m u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \, \mathscr{D}^\alpha g_\alpha(x), \tag{55}$$

where $g_\alpha \in W^{1,2}(\Omega)$ and

$$\|g_\alpha\|_{1,2} \leqslant C_{10} \sum_{|\gamma| \leqslant m-1} \|h_\gamma\|_2, \qquad |\alpha| \leqslant m. \tag{56}$$

Applying [8, Chapter 7, Theorems 3, 4] to the solution $u_h(x)$ of (55), we obtain that $u_h \in W^{m+1,2}(\Omega)$ and

$$\|u_h\|_{m+1,2} \leqslant C_{11} \sum_{|\gamma| \leqslant m-1} \|h_\gamma\|_2 \leqslant C_{12} \, \|h\|_*^{(2)}. \tag{57}$$

Therefore, $u_h \in D(A)$ and from the definition of the operator A follows that $u_h(x)$ is the solution of the equation (52).

From the inequality (51) we have that the operator A is one-to-one. Thus, the inverse operator $A^{-1}\colon [X^{(2)}]^* \to D(A)$ transforms the element $h \in [X^{(2)}]^*$ to $u_h \in D(A)$ which is the solution of the equation (52). From the inequality (51) and the compactness of the embedding $W^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$, it follows that the operator A^{-1} is compact. ∎

Proof of Theorem 7. We introduce the operator $C\colon \overline{\mathscr{R}_+} \times X^{(2)} \to [X^{(2)}]^*$ by

$$\langle C(\lambda, u), \phi \rangle = \sum_{|\gamma| \leqslant m-1} \int_\Omega C_\gamma(\lambda, x, ..., \mathscr{D}^{m-1} u) \, \mathscr{D}^\gamma \phi(x) \, dx, \tag{58}$$

and shall verify that this operator C as well as the operator A, defined by (50), satisfies all the conditions of Theorem 4. Thus, the assertion of Theorem 7 will follow from Theorem 4. We fix the set $\mathscr{D}$ which is as in the statement of Theorem 7.

From the proof of Lemma 3 it follows that A is a maximal monotone operator with compact resolvent $(A+J)^{-1}$. It is necessary to check that the operator A satisfies condition $(A_\infty^{(0)})$ on $\bar{\mathscr{D}} \cap D(A)$. Let $h_n \in [X^{(2)}]^*$ be such that

$$Au_n = h_n, \qquad u_n \in D(A) \cap \bar{\mathscr{D}}, \quad \|h_n\|_*^{(2)} \to \infty. \tag{59}$$

From the equality $\langle Au_n, u_n\rangle = \langle h_n, u_n\rangle$, the boundedness of the set $\bar{\mathscr{D}}$ in $X^{(2)}$ and the estimate (51), we obtain the inequality

$$\|u_n\|_{m,2}^2 \leqslant C_{13} \|h_n\|_*^{(2)}, \tag{60}$$

where the constant C_{13} is independent of h_n. Passing to a subsequence, if necessary, we may assume that $\overline{h_n} \equiv (\|h_n\|_*^{(2)})^{-1} \cdot h_n \rightharpoonup \bar{h}$. It is necessary to prove only that $\bar{h} = 0$. To this end, let $\phi \in \mathscr{C}_0^\infty(\Omega)$ be a given function. Then, by condition $(\mathscr{A}_2^{(2)})$,

$$\begin{aligned}
&[\|h_n\|_*^{(2)}]^{-1} \cdot |\langle Au_n, \phi\rangle| \\
&\quad = \frac{1}{\|h_n\|_*^{(2)}} \left| \int_\Omega \sum_{|\alpha|=m} A_\alpha(x, \mathscr{D}^m u_n)\, \mathscr{D}^\alpha \phi(x)\, dx \right| \\
&\quad \leqslant \frac{C_{14}}{\|h_n\|_*^{(2)}} \left\{ \int_\Omega \left[1 + \sum_{|\alpha|=m} |\mathscr{D}^\alpha u_n(x)| \right]^2 dx \right\}^{1/2} \left\{ \sum_{|\alpha|=m} \int_\Omega |\mathscr{D}^\alpha \phi(x)|^2\, dx \right\}^{1/2}.
\end{aligned}$$

Using the inequality (60), we obtain that the right-hand side of the last inequality tends to zero as $n \to \infty$. For the same function $\phi(x)$, we also have from (59)

$$0 = \lim_{n\to\infty} \left\langle \frac{1}{\|h_n\|_*^{(2)}} Au_n, \phi \right\rangle = \lim_{n\to\infty} \langle \overline{h_n}, \phi\rangle = \langle \bar{h}, \phi\rangle$$

and, consequently, $\bar{h} = 0$. We have shown that the operator A satisfies condition $(A_\infty^{(0)})$.

Now, we shall check that the operator C, defined by (58), satisfies the conditions of Theorem 4. From conditions $(\mathscr{C}_1^{(2)})$, $(\mathscr{C}_2^{(2)})$ on the functions $C_\gamma(\lambda, x, \xi^{(0)}, ..., \xi^{(m-1)})$ it follows that the operator C is bounded and continuous. For this operator we have the estimate

$$\|C(\lambda, u)\|_*^{(2)} \leqslant C_{15} f(\lambda), \tag{61}$$

for $u \in \bar{\mathscr{D}}$ with constant C_{15} independent of λ and u.

Let u be an arbitrary function from $\bar{\mathscr{D}}$. Given a number $\varepsilon > 0$, we can choose a number $M_\varepsilon > 0$, depending only on ε, and a set $\Omega_\varepsilon \subset \Omega$ depending on u and ε such that

$$mes(\Omega \backslash \Omega_\varepsilon) < \varepsilon, \qquad |\mathscr{D}^\gamma u(x)| \leqslant M_\varepsilon, \qquad \text{for} \quad x \in \overline{\Omega_\varepsilon},$$

$$\int_{\Omega \backslash \Omega_\varepsilon} |\mathscr{D}^\gamma u(x)|\, dx < \varepsilon, \qquad \text{for} \quad |\gamma| \leqslant m-1. \tag{62}$$

From the conditions $(\mathscr{C}_2^{(2)})$, $(\mathscr{C}_3^{(2)})$ we have

$$\begin{aligned}
&\sum_{|\gamma| \leqslant m-1} \int_\Omega C_\gamma(\lambda, x, u(x), ..., \mathscr{D}^{m-1}u(x))\, \mathscr{D}^\gamma \omega(x)\, dx \\
&\quad = f(\lambda) \sum_{|\gamma| \leqslant m-1} \int_{\Omega_\varepsilon} [\, C'_\gamma(x, u(x), ..., \mathscr{D}^{m-1}u(x)) + R_{\gamma, \lambda}(\lambda, x)]\, \mathscr{D}^\gamma \omega(x)\, dx \\
&\qquad + \sum_{|\gamma| \leqslant m-1} \int_{\Omega \backslash \Omega_\varepsilon} C_\gamma(\lambda, x, u(x), ..., \mathscr{D}^{m-1}u(x))\, \mathscr{D}^\gamma \omega(x)\, dx \\
&\quad \geqslant f(\lambda) \left\{ \int_{\Omega_\varepsilon} C''(x)\, dx + \int_{\Omega_\varepsilon} \sum_{|\gamma| \leqslant m-1} R_{\gamma, u}(\lambda, x)\, \mathscr{D}^\gamma \omega(x)\, dx \right. \\
&\qquad \left. - \sum_{|\gamma| \leqslant m-1} \int_{\Omega \backslash \Omega_\varepsilon} \left[1 + \sum_{j=0}^{m-1} |\mathscr{D}^j u(x)| \right] |\mathscr{D}^\gamma \omega(x)|\, dx \right\}.
\end{aligned} \tag{63}$$

Here,

$$\sup_{u \in \bar{\mathscr{D}}} \{ \sup_{x \in \Omega_\varepsilon} |R_{\gamma, u}(\lambda, x)| \} \to 0 \qquad \text{as} \quad \lambda \to \infty$$

and $\omega(x)$ is the same function as in condition $(\mathscr{C}_3^{(2)})$.

Taking into consideration the fact that ε is an arbitrarily small positive number, we obtain from (62), (63) that for every function $u \in \bar{\mathscr{D}}$ we have the inequality

$$\sum_{|\gamma| \leqslant m-1} \int_\Omega C_\gamma(\lambda, x, u(x), ..., \mathscr{D}^{m-1}u(x))\, \mathscr{D}^\gamma \omega(x) \geqslant C_{16} f(\lambda), \tag{64}$$

where $C_{16} > 0$ if $\lambda \geqslant \Lambda$, for some $\Lambda > 0$ and independent of u. From (64) we have

$$\| C(\lambda, u) \|_*^{(2)} \geqslant C_{17} f(\lambda) \qquad \text{for} \quad \lambda \geqslant \Lambda, \quad u \in \bar{\mathscr{D}}, \tag{65}$$

with $C_{17} > 0$, and, consequently, the operator C satisfies condition $(\mathscr{C}_2)$ of Theorem 4.

To check condition $(\mathscr{C}_1)$ of Theorem 4, we let $\{\lambda_n\}$, $\{u_n\}$ be such that

$$\lambda_n \geqslant \Lambda, \qquad u_n \in \bar{\mathscr{D}}, \qquad \|Ju_n + Au_n\|_*^{(2)} \leqslant 2C_{15} f(\lambda_n).$$

If the sequence $\{\lambda_n\}$ is bounded, then Lemma 3 implies the compactness of the sequence $\{u_n\}$ in $X^{(2)}$. This allows us to assume the $u_n(x) \to u_0(x)$ in $X^{(2)}$. It is easy to see that in this case the sequence

$$Q_n \equiv [\|C(\lambda_, u_n)\|_*^{(2)}]^{-1} C(\lambda_n, u_n)$$

is strongly convergent to some $h \neq 0$.

Consider the case $\lambda_n \to \infty$. From (61) and (65) we have

$$C_{17} f(\lambda_n) \leqslant \|C(\lambda_n, u_n\|_*^{(2)} \leqslant C_{15} f(\lambda_n)$$

and from the inequality (64) we have the estimate

$$\langle [\|C(\lambda_n, u_n)\|_*^{(2)}]^{-1} \cdot C(\lambda_n, u_n), \omega \rangle \geqslant \frac{C_{16}}{C_{15}} > 0.$$

This estimate guarantees that the weak limit of the sequence Q_n cannot equal to zero, and establishes the fact that the operator C satisfies condition $(\mathscr{C}_1)$ of Theorem 4. This finishes the proof of Theorem 7. ∎

Problem 3. We assume that the functions $a_i(x, p)$, $i = 1, ..., n$, $p = (p_1, ..., p_n) \in \mathscr{R}^n$, satisfy the following conditions:

$(\mathscr{A}_1^{(3)})$ $a_i(x, p)$ is defined and differentiable w.r.t. all of its arguments for $x \in \bar{\Omega}$, $p = (p_1, ..., p_n) \in \mathscr{R}^n$, where Ω is a bounded open set in $\mathscr{R}^n$ with boundary $\partial\Omega$ of class $\mathscr{C}^2$. Moreover, $a_i(x, 0) \equiv 0$ for $i = 1, ..., n$, $x \in \bar{\Omega}$;

$(\mathscr{A}_2^{(3)})$ there exist constants K_3, K_4 such that the inequalities

$$\begin{gathered}\sum_{i,j=1}^{n} \frac{\partial a_i(x, p)}{\partial p_j} \xi_i \xi_j \geqslant K_3(1 + |p|)^{m-2} \cdot \sum_{i=1}^{n} \xi_i^2, \\ \left|\frac{\partial a_i(x, p)}{\partial p_j}\right| \leqslant K_4(1 + |p|)^{m-2}, \qquad \left|\frac{\partial a_i(x, p)}{\partial x_k}\right| \leqslant K_4(1 + |p|)^{m-1}\end{gathered} \tag{66}$$

are satisfied.

Assume that the function $C(\lambda, x, u)$ satisfies the following conditions:

($\mathscr{C}_1^{(3)}$) $C(\lambda, x, u)$ is defines and continuous for $(\lambda, x, u) \in \overline{\mathscr{R}_+} \times \bar{\Omega} \times \mathscr{R}$ and $C(0, x, u) \equiv 0$;

($\mathscr{C}_2^{(3)}$) there exists a function $f(\lambda)$ and a continuous function $C'(x, u)$, defined for $(x, u) \in \bar{\Omega} \times \mathscr{R}$, such that

$$|C(\lambda, x, u)| \leqslant f(\lambda)(1+|u|), \qquad f(\lambda) \geqslant 1,$$
$$\lim_{\lambda \to \infty} f(\lambda) = +\infty, \qquad \lim_{\lambda \to \infty} \frac{1}{f(\lambda)} C(\lambda, x, u) = C'(x, u), \tag{67}$$

for $(\lambda, x, u) \in \overline{\mathscr{R}_+} \times \bar{\Omega} \times \mathscr{R}$, where the second limit is uniform w.r.t. x, u on an arbitrary bounded set;

($\mathscr{C}_3^{(3)}$) there exists a continuous function $C''(x)$, $x \in \bar{\Omega}$, such that the inequalities

$$C'(x, u) \geqslant C''(x), \qquad \int_{\Omega} C''(x)\, dx > 0 \tag{68}$$

are valid for $x \in \bar{\Omega}$, $u \in \mathscr{R}$.

THEOREM 8. *Assume that the functions $a_i(x, p)$, $C(\lambda, x, u)$, $i = 1, ..., n$, satisfy the conditions ($\mathscr{A}_1^{(3)}$), ($\mathscr{A}_2^{(3)}$) and ($\mathscr{C}_1^{(3)}$)–($C_3^{(3)}$), respectively. Then for an arbitrary bounded set $\mathscr{D} \subset L^2(\Omega)$, such that $0 \in \mathscr{D}$, there exist $\lambda_0 \in \mathscr{R}_+$, $u_0 \in \partial\mathscr{D}$ satisfying the eigenvalue problem*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u_0(x)}{\partial x}\right) + C(\lambda_0, x, u_0(x)) = 0, \qquad u_0 \in W_0^{1, m}(\Omega). \tag{69}$$

Proof. We shall reduce the eigenvalue problem (69) to an eigenvalue problem for an operator equation with operators satisfying the conditions of Theorem 5 in the space $X^{(3)} = L^2(\Omega)$.

We introduce the operators

$$A\colon X^{(3)} \supset D(A) \to X^{(3)}, \qquad C\colon \overline{\mathscr{R}_+} \times X^{(3)} \to X^{(3)}$$

by the equalities

$$(Au)(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u(x)}{\partial x}\right), \qquad (C(\lambda, u))(x) = C(\lambda, x, u(x)), \tag{70}$$

where

$$D(A)=\left\{u\in X^{(3)}: u\in W_0^{1,m}(\Omega), \left[1+\left|\frac{\partial u}{\partial x}\right|\right]^{m-2}\left|\frac{\partial^2 u(x)}{\partial x^2}\right|\in L^2(\Omega)\right\}.$$

We shall first prove that the operator A, defined by (70), is an m-accretive operator acting from $D(A)$ onto $X^{(3)}$. Given $u, v\in D(A)$, we have the inequality

$$\langle Au-Av, u-v\rangle \geqslant C_{18}\int_\Omega \left|\frac{\partial(u-v)}{\partial x}\right|^m dx, \tag{71}$$

which follows from the first inequality in (66). Here, $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$. Therefore, the operator A is accretive.

Let h be an arbitrary function in $X^{(3)}$ and define the function $w\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ as a solution of the equation $\Delta w(x)=h(x)$, where Δ is the Laplacian operator. Consider the boundary value problem

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right)=\sum_{i=1}^n \frac{\partial f_i(x)}{\partial x}, \qquad u\in W_0^{1,m}(\Omega), \tag{72}$$

where $f_i(x)=\partial w(x)/\partial x_i$. From the theory of monotone operators, it is a simple matter to prove the existence of a solution of the problem (72), which we denote by $u_h(x)$. For this solution we have the estimate

$$\|u_h\|_{1,m}^{m-1}\leqslant C_{13}\,\|h\|^{(3)}, \tag{73}$$

where $\|\cdot\|_{1,m}$, $\|\cdot\|^{(3)}$ are norms on $W_0^{1,m}(\Omega)$ and $X^{(3)}$, respectively. We shall prove that $u_h\in D(A)$.

Using [8, Chapter 7, Theorems 3.1, 3.3], we see that it is possible to obtain the inclusion

$$\left(1+\left|\frac{\partial u_h(\cdot)}{\partial x}\right|\right)^{(m-2)/2}\left|\frac{\partial^2 u_h(\cdot)}{\partial x^2}\right|\in L^2(\Omega). \tag{74}$$

We note that [8, Chapter 7, Theorem 3.3] implies the local inclusion of type (74) (near the boundary of $\partial\Omega$) for all second derivatives except $\partial^2 u(x)/\partial n_x^2$, where n_x is the direction of the normal to the boundary of Ω at the point x. However, the term with derivative $\partial^2 u(x)/\partial n_x^2$ can be expressed via other second order derivatives by using Eq. (72).

Starting from (74), it is possible to prove in finite number of steps that the inclusions

$$\left(1+\left|\frac{\partial u_h(\cdot)}{\partial x}\right|\right)^{q_j}\left|\frac{\partial^2 u_h(\cdot)}{\partial x^2}\right| \in L^2(\Omega), \qquad j=1, ..., J, \tag{75}$$

where

$$\frac{m-2}{2}=q_1<q_2<\cdots<q_J=m-2.$$

The inclusions in (75) follow from some local estimates which are obtained from the integral identity corresponding to the problem (72) after the substitutions of special test functions. For example, to obtain local interior inclusions of the type (75), it is necessary to substitute in the integral identity the functions

$$\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left[T_M\left(\left|\frac{\partial u_h(x)}{\partial x}\right|^{\beta_j}\right)\frac{\partial u_h(x)}{\partial x_k}\cdot\psi^2(x)\right], \qquad j=1, ..., J,$$

where

$$0<\beta_j\leqslant\frac{m-2}{2},$$

$T_M(t)$ is a smooth nondecreasing function such that $T_M(t)=t$ for $t\leqslant M$ and $T_M(t)=M+1$ for $t\geqslant M+1$, and $\psi(x)$ is a cut-off function. In this way, we can prove the inclusion (75) for $q_J=m-2$ and, subsequently, establish the fact that $u_h\in D(A)$.

We have shown that $R(A)=X^{(3)}$ for the operator A defined by (70). From the inequality (71) follows that A is a one-to-one operator. Thus, the inverse operator $A^{-1}\colon X^{(3)}\to D(A)$ exists. From the estimate (71) we also obtain that A^{-1} is compact. It is easy to see that the operator $(I+A)^{-1}$ has the same properties as A^{-1}.

We shall now verify that the operators A, C, defined by (70), satisfy all the conditions of Theorem 5. At first, we shall establish that the operator A satisfies the condition $(A_\infty^{(0)})$ on $D(A)\cap\bar{\mathscr{D}}$, where $\mathscr{D}$ is some fixed set satisfying the assumptions of Theorem 8. Let $\{h_n\}\subset X^{(3)}$ be such that

$$Au_n=h_n, \qquad u_n\in D(A)\cap\bar{\mathscr{D}}, \quad \|h_n\|^{(3)}\to\infty. \tag{76}$$

Using the equality $\langle Au_n, u_n\rangle=\langle h_n, u_n\rangle$, the estimate (71), and the boundedness of the set $\bar{\mathscr{D}}$, we obtain the inequality

$$\|u_n\|_{1,m}^{m}\leqslant C_{20}\,\|h_n\|^{(3)}, \tag{77}$$

for some constant C_{20} independent of n. Passing to a subsequence, if necessary, we may assume that

$$\overline{h_n} = [\,\|h_n\|^{(3)}]^{-1} \cdot h_n \rightharpoonup \overline{h_0} \in X^{(3)}.$$

It is necessary to show that $\bar{h}=0$. Given $\phi \in \mathscr{C}_0^\infty(\Omega)$, we have the estimate

$$\begin{aligned}
&\langle [\,\|h_n\|^{(3)}]^{-1} Au_n, \phi(x)\rangle \\
&\quad = [\,\|h_n\|^{(3)}]^{-1} \int_\Omega \sum_{i=1}^n a_i\left(x, \frac{\partial u_n(x)}{\partial x}\right) \frac{\partial \phi(x)}{\partial x_i}\, dx \\
&\quad \leqslant C_{21}[\,\|h_n\|^{(3)}]^{-1} \left\{\int_\Omega \left(1+\left|\frac{\partial u_n(x)}{\partial x}\right|\right)^m dx\right\}^{(m-1)/m} \cdot \left\{\int_\Omega \left|\frac{\partial \phi(x)}{\partial x}\right|^m dx\right\}^{1/m}.
\end{aligned} \tag{78}$$

The right-hand side of (78) tends to zero by virtue of (76), (77). From (76) and (78) we obtain that $\bar{h}=0$. Consequently, the operator A satisfies the condition $(A_\infty^{(0)})$.

Let us now check that the operator C from (70) satisfies the conditions of Theorem 5. In view of the properties of the Nemytskij operator and conditions $(\mathscr{C}_1^{(3)})$, $(\mathscr{C}_2^{(3)})$, we have the boundedness and the continuity of the operator C. We also have the estimate

$$\|C(\lambda, u)\|^{(3)} \leqslant C_{22} f(\lambda), \qquad \text{for} \quad u \in \overline{\mathscr{D}}, \tag{79}$$

with the constant C_{22} independent of λ, u.

Let $u \in \overline{\mathscr{D}}$ and $\varepsilon > 0$ be arbitrary. It is possible to choose a number M_ε (depending only on ε) and a set $\Omega_\varepsilon \subset \Omega$ (depending on u, ε) such that the following inequalities are valid:

$$\begin{gathered}
mes(\Omega\backslash\Omega_\varepsilon) < \varepsilon, \qquad |u(x)| \leqslant M_\varepsilon, \qquad \text{for} \quad x \in \Omega_\varepsilon, \\
\int_{\Omega\backslash\Omega_\varepsilon} |u(x)|\, dx < \varepsilon.
\end{gathered} \tag{80}$$

From the conditions $(\mathscr{C}_2^{(3)})$, $(\mathscr{C}_3^{(3)})$ we have

$$\begin{aligned}
&\int_\Omega C(\lambda, x, u(x))\, dx \\
&\quad = \int_{\Omega\backslash\Omega_\varepsilon} C(\lambda, x, u(x))\, dx + f(\lambda) \int_{\Omega_\varepsilon} [\, C'(x, u(x)) + R_u(\lambda, x)]\, dx \\
&\quad \geqslant f(\lambda) \left\{\int_{\Omega_\varepsilon} C''(x)\, dx - \int_{\Omega_\varepsilon} |R_u(\lambda, x)|\, dx - C_{23} \int_{\Omega\backslash\Omega_\varepsilon} [1+|u(x)|]\, dx\right\}.
\end{aligned} \tag{81}$$

Here, $R_u(\lambda, x)$ is such that

$$\sup_{u \in \overline{\mathscr{D}}} \{ \sup_{x \in \Omega_\varepsilon} |R_u(\lambda, x)| \} \to 0 \qquad \text{as} \quad \lambda \to \infty. \tag{82}$$

Taking into consideration the inequalities (68), (80), and (82), we obtain from (81), by a suitable choice of ε, the estimate

$$\int_\Omega C(\lambda, x, u(x))\, dx \geqslant C_{24} f(\lambda), \qquad \text{for} \quad \lambda \geqslant \Lambda, \tag{83}$$

where C_{24} is a positive constant, Λ is sufficiently large, and u is an arbitrary function from $\overline{\mathscr{D}}$.

Inequality (83) implies that the operator C satisfies condition $(\mathscr{C}_2)$ of Theorem 5. To check that condition $(\mathscr{C}_1)$ of Theorem 5 holds, let $\{\lambda_n\}$, $\{u_n\}$ be such that

$$\lambda_n \geqslant \Lambda, \qquad u_n \in \overline{\mathscr{D}}, \qquad \|u_n + Au_n\|^{(3)} \leqslant 2C_{22} f(\lambda_n).$$

If the sequence $\{\lambda_n\}$ is bounded, then from the estimate (71) we have the compactness of the sequence $\{u_n\}$ in $X^{(3)}$. In this case the set

$$\{[\|C(\lambda_n, u_n)\|^{(3)}]^{-1} \cdot C(\lambda_n, u_n)\}$$

is compact and the weak closure of this set does not contain zero.

If, on the other hand, $\lambda_n \to \infty$, then we have from (79), (83) the estimate

$$C_{25} f(\lambda_n) \leqslant \|C(\lambda_n, u_n)\|^{(3)} \leqslant C_{22} f(\lambda_n),$$

and from the inequality (83) we obtain

$$\int_\Omega [\|C(\lambda_n, u_n)\|^{(3)}]^{-1} \cdot C(\lambda_n, x, u_n(x))\, dx \geqslant \frac{C_{24}}{C_{22}} > 0.$$

Consequently, the weak limit of the sequence

$$\{[\|C(\lambda_n, u_n)\|^{(3)}]^{-1} \cdot C(\lambda_n, u_n)\}$$

cannot equal zero. We have shown that the operator C satisfies condition (C_1) of Theorem 5. The proof is complete. ∎

5. A RESULT WITH C DEFINED ONLY ON THE DOMAIN OF A

In this section we demonstrate the possibility to improve Theorem 10 of Kartsatos [4] where the operator C may be defined only on the domain of the operator A.

Given a maximal monotone operator $A: X \supset D(A) \to 2^{X^*}$, $x \in X$ and $\mu > 0$, there exists a unique $x_\mu \in D(A)$ such that

$$J(x_\mu - x) + \mu A x_\mu \ni 0.$$

We set

$$J_\mu x = x_\mu \qquad \text{and} \qquad A_\mu x = \frac{1}{\mu} J(x - x_\mu).$$

We know that $J_\mu: X \to X$, $A_\mu: X \to X^*$ are single-valued and bounded operators. We also know that both operators are demicontinuous and bounded with A_μ maximal monotone. In addition, it is easy to see that $A_\mu x \in A J_\mu x$, $x \in X$.

In order to solve the eigenvalue problem

$$Au - C(\lambda, u) \ni 0, \tag{84}$$

is suffices to solve the problem

$$A_\mu x - C(\lambda, J_\mu x) = 0, \tag{85}$$

for some $\mu > 0$.

THEOREM 9. *Let X, X^* be locally uniformly convex with X reflexive. Let $\mathscr{D}$ be a bounded open subset of X. Assume that $A: X \supset D(A) \to 2^{X^*}$ is maximal monotone and such that $0 \in D(A) \cap \mathscr{D}$, $0 \in A(0)$ and $0 \notin A(J_\mu(\partial \mathscr{D}))$, for some $\mu > 0$. Assume that $(J + A)^{-1}$ is compact. Let $C: \mathscr{R}_+ \times D(A) \to X^*$ and assume that the operator $(\lambda, x) \to C(\lambda, J_\mu x)$ is compact and satisfies conditions* (i) *and* (ii) *of Lemma* 1. *Moreover, $C(0, u) = 0$, $u \in D(A)$. Then there exists $(\lambda_0, u_0) \in \mathscr{R}_+ \times D(A)$ such that $Au_0 - C(\lambda_0, u_0) \ni 0$.*

Proof. We know that the operator J_μ is compact. In fact, this follows from the compactness of the operator J_1 (cf. [4, Lemma 3]). By Lemma 1, there exists a compact operator $\tilde{C}: \overline{\mathscr{R}_+ \times \bar{\mathscr{D}}} \to X^*$ such that

$$\tilde{C}(\lambda, x) = C(\lambda, J_\mu x), \qquad (\lambda, x) \in \overline{\mathscr{R}_+} \times \partial \mathscr{D},$$

and

$$\lim_{\lambda \to \infty} \mu_\lambda = +\infty, \qquad \text{where} \quad \mu_\lambda = \inf\{\|\tilde{C}(\lambda, x)\| : x \in \bar{\mathscr{D}}\}. \tag{86}$$

In order to solve first the problem (85), we consider the homotopy equation

$$H(\lambda, x) \equiv x - J_\mu x - \mu J^{-1} \tilde{C}(\lambda, x) = 0. \tag{87}$$

The function $H: \mathscr{R}_+ \times \bar{\mathscr{D}} \to X$ is a compact displacement of the identity. Equation (87) is equivalent to the equation

$$A_\mu x - \tilde{C}(\lambda, x) = 0.$$

We want to show that there exists $\bar{\lambda} \in (0, \infty)$ such that $deg(H(\bar{\lambda}, \cdot), \bar{D}, 0) = 0$. Assume that this is not true. Then, by the Leray–Schauder theory, there exist sequences $\{\lambda_n\} \subset \mathscr{R}_+$ and $\{x_n\} \subset \bar{\mathscr{D}}$ such that $\lambda_n \to \infty$ and

$$A_\mu x_n - \tilde{C}(\lambda_n, x_n) = 0.$$

This, along with (86), implies

$$\lim_{n\to\infty} \|A_\mu x_n\| = +\infty,$$

i.e., a contradiction to the boundedness of the operator A_μ.

Now, we consider the homotopy equation

$$H_2(t, x) \equiv x - tJ_\mu x = 0, \qquad (t, x) \in [0, 1] \times \bar{\mathscr{D}}.$$

If we assume that there is a solution $x_t \in \partial\mathscr{D}$ of this equation, then we get, exactly as in [4], a contradiction to the condition $0 \notin A(J_\mu(\partial\mathscr{D}))$. Thus,

$$deg(H_2(t, \cdot), \bar{\mathscr{D}}, 0) = deg(H_2(0, \cdot), \bar{\mathscr{D}}, 0) = deg(I, \bar{\mathscr{D}}, 0) = 1.$$

However, $H(0, x) = H_2(1, x)$, $x \in \bar{\mathscr{D}}$. Thus,

$$deg(H(0, \cdot), \bar{\mathscr{D}}, 0) = 1.$$

It follows that there is $\lambda_0 \in \overline{\mathscr{R}_+}$ and $x_0 \in \partial\mathscr{D}$ such that

$$A_\mu x_0 - \tilde{C}(\lambda_0, x_0) = 0.$$

We cannot have $\lambda_0 = 0$ because we already know that $H_2(1, x) = 0$ has no solution on $\partial\mathscr{D}$. Since $x_0 \in \partial\mathscr{D}$, we have $C(\lambda_0, J_\mu x_0) = \tilde{C}(\lambda_0, x_0)$. Letting $u_0 = J_\mu x_0 \in D(A)$ and observing that $A_\mu u_0 \in AJ_\mu u_0$, we obtain $Au_0 - C(\lambda_0, u_0) \ni 0$. ∎

Remark 6. Repeating the argument of [4, p. 1696], if

$$L \equiv \sup_{x\in\bar{\mathscr{D}}} \{\|x\|\},$$

then

$$\|J_\mu x\| \leqslant 2\,\|x\| \leqslant 2L, \qquad x \in \bar{\mathscr{D}}.$$

This means that the assumptions on C in Theorem 9 can be replaced by the following: $C\colon \overline{\mathscr{R}_+} \times D(A) \to X^*$ is such that $C(0, u) = 0$, $u \in D(A)$, $x \to C(\lambda, J_\mu x)$ is compact w.r.t. its second variable and

$$\lim_{\lambda \to \infty} \mu_\lambda = +\infty, \qquad \text{where} \quad \mu_\lambda = \inf\{ \|C(\lambda, u)\| : u \in D(A) \cap \overline{B_{2L}(0)} \}.$$

Then we have

$$\inf\{ \|C(\lambda, J_\mu x)\| : x \in \overline{\mathscr{D}} \} \geqslant \inf\{ \|C(\lambda, u)\| : u \in D(A) \cap \overline{B_{2L}(0)} \} = \mu_\lambda$$

and can work in the proof of Theorem 9 directly with the operator C instead of $\tilde{C}$.

Naturally, a result like Theorem 9 also holds for the case of m-accretive operators A. Moreover, several results in Section 3 remain true, under the appropriate assumptions, for multi-valued operators A.

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